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Source-sink flow in a rotating cylinder

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Summary

This paper describes the axisymmetric source-sink flow in a rapidly rotating cylinder. Relative fluid motion is induced by the presence of a sink in the bottom corner and a ring source located somewhere in the fluid, at some distance from the solid boundaries. In order to neglect nonlinear effects the volumetric flow rates are assumed to be small, i.e. $O(E^{1/2})$, with E the Ekman number of the flow. The transport from the source to the sink is carried by Ekman layers at the end caps, and a Stewartson layer at the sidewall. At the ring source a free Stewartson layer arises, in which the injected fluid is transported towards the Ekman layers. This Stewartson layer consists of layers of thicknesses $E^{1/4}$ and $E^{1/3}$, which both contribute to the vertical $O(E^{1/2})$ transport. The ring source is enveloped by a ring-shaped region of cross-sectional dimensions $O(E^{1/2} \times E^{1/2})$, in which the injected fluid is rearranged before erupting into the $E^{1/3}$ layer. As $E^{1/2} \ll E^{1/3}$, this region appears as an isolated singularity in the $E^{1/3}$ layer; in fact it consists of a combination of an upward and a downward directed source, the strengths of which can be determined by transport arguments. The paper presents an analysis of the $E^{1/3}$-layer structure on the basis of a linear theory; it also describes how the analysis can be extended to the situation in which fluid is injected through an array of sources at different heights.

1. Introduction

Partly because of geophysical and engineering interests, several investigators (e.g. [1] to [5]) have studied source-sink flows in rapidly rotating fluids in a variety of configurations. An important engineering application of such flows concerns the gas centrifuge, as used for separation of uranium isotopes (see for example [6], [7]). Although gas flows are essentially affected by compressibility effects, studies of incompressible fluids in similar centrifuge configurations are helpful to obtain insight in the complicated flow characteristics. For mathematical convenience most investigators restricted themselves to axisymmetric source-sink arrangements, besides assuming the relative velocities to be small enough as to neglect inertial effects.

Hashimoto [4] has investigated the flow of an incompressible fluid in a rapidly rotating cylinder due to line sources and sinks distributed along concentric circles on the top and bottom disks. As a result of these axisymmetric source-sink distributions free vertical shear layers arise in the otherwise geostrophically controlled flow. Their structure is identical to the shear layers studied by Stewartson [8] and consists of layers of thicknesses $E^{1/4}$ and $E^{1/3}$ (E is the Ekman number of the flow). In combination with the Ekman layers at the horizontal end caps the Stewartson layers play an essential part in carrying fluid from the sources to the sinks.

A number of interesting source-sink arrangements has been studied experimentally by

Hide [2]. In the simplest configuration considered, fluid was confined in a rapidly rotating annular region between two concentric cylinders and two horizontal top and bottom planes. Relative motion was generated by applying uniform injection and uniform withdrawal at the inner and outer sidewalls, respectively. Hide showed that the injected fluid is entirely carried by a Stewartson layer at the inner sidewall, the Ekman layers at the horizontal planes, and a Stewartson layer at the outer sidewall of the annulus; the flow outside the boundary layers is in geostrophic balance, and is governed by the Ekman layers.

In a recent theoretical study Conlisk and Walker [9] investigated the fluid flow in a similar configuration, now due to uniform injection and withdrawal through thin axisymmetric slots in the sidewalls; they considered two different slot sizes, viz. $O(E^{1/2})$ and $O(1)$, the latter value corresponding with a distributed source or sink. Special attention was given to the Stewartson layers at the sidewalls of the annulus. As in Hide's configuration [2] these shear layers play an essential part in transporting fluid from the sources to the sinks. As a consequence of its quasi-geostrophic character, the $E^{1/4}$ layer proves to be independent of the method of injection or withdrawal; details of the injection mode are only noticeable in the inner $E^{1/3}$ layer.

The present paper focusses on the axisymmetric flow driven by injection from a ring source that is located at some distance from the solid flow boundaries, and withdrawal from a ring sink at the sidewall. When the source and the sink are not at the same radius, the transport from the source to the sink is carried by Ekman layers at the horizontal planes. Therefore, a free Stewartson shear-layer will arise at the line source, in order to carry the injected fluid to the Ekman layers. As in other configurations, [3,4,8,9], this Stewartson layer has a sandwich structure consisting of an outer $E^{1/4}$ layer and an inner $E^{1/3}$ layer, with the former also playing a part in the matching of the geostrophic swirl velocities on either side of the shear layer. The ring source, which is assumed to have an infinitesimal cross-section, is enveloped by a ring-shaped region of cross-sectional dimensions $O(E^{1/2} \times E^{1/2})$ in which the injected fluid is redistributed before it erupts into the $E^{1/3}$ layer. Because of its small lateral dimensions such a region appears as an isolated singularity in the $E^{1/3}$ layer, and can be described by a pair of upward and downward directed sources. It is shown in this paper that the individual source strengths can be determined from external transport requirements.

A general formulation of the flow problem and a brief description of the geostrophic flow, the Ekman layers, and the Stewartson layer at the sidewall will be presented in Section 2. The Stewartson layer arising at the ring source is analysed in Section 3, and a discussion of the main results follows in Section 4.

2. Formulation

In order to formulate the problem mathematically, consider the flow of an incompressible fluid enclosed by two horizontal disks at distance HL apart, and a cylinder with radius bL (see Fig. 1). This system rotates at angular velocity Ω about the (vertical) axis of symmetry. A steady relative motion of the fluid is generated by axisymmetric injection at a flow rate Q^* through a ring-shaped line source of infinitesimal cross-sectional dimensions at some height hL ($< HL$); the radius of the ring is aL ($< bL$) and its symmetry axis coincides with the rotation axis. Fluid is withdrawn at the sidewall through a ring-shaped line source in the bottom corner.

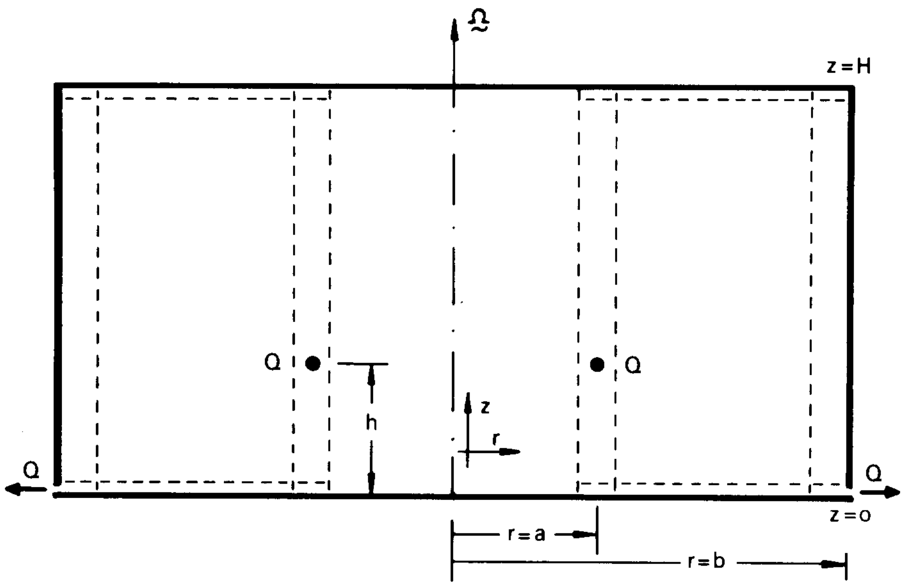

Figure 1. Schematic diagram of the configuration; fluid is emitted at flow rate Q from a ring source located at $(r = a, z = h)$, and withdrawn at rate Q through a sink at $(r = b, z = 0)$.

For a convenient mathematical analysis of the flow lengths and velocities are nondimensionalized by the length scale L and some characteristic velocity U^*, respectively. When referred to a frame that rotates steadily with angular velocity Ω, the dimensionless equations of momentum and conservation of mass for stationary fluid flow are:

$$\mathrm{Ro}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + 2\boldsymbol{k}\times\boldsymbol{v} = -\nabla p + E\nabla^2\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v} = 0 \tag{2.1}$$

with $\boldsymbol{v}$ the velocity vector in the rotating frame, p the reduced pressure and $\boldsymbol{k}$ a unit vector in axial direction: $\boldsymbol{k} = \boldsymbol{\Omega}/\Omega$. The two dimensionless parameters in (2.1) are the Rossby number Ro and the Ekman number E, defined as

$$\mathrm{Ro} = U^*/\Omega L, \quad E = \nu/\Omega L^2, \tag{2.2}$$

where ν is the kinematic fluid viscosity. It is assumed that the source-sink driven fluid motion is slow enough as to neglect the non-linear terms in (2.1), which assumption requires $\mathrm{Ro} \ll E^{1/4}$ for axisymmetric flows (see [5]). In addition it is assumed that the container is rapidly rotating, i.e. $E \ll 1$. In order to have a small Rossby-number flow, attention is restricted to small flow rates

$$Q^* = QU^*L^2E^{1/2}, \tag{2.3}$$

where Q is the dimensionless volumetric flow rate.

Throughout this paper the relative fluid motion is related to a co-rotating cylindrical coordinate system (r, θ, z), with the z-axis coinciding with the rotation axis; radial, azimuthal and axial velocity components are denoted by (u, v, w). The bottom and top disks of the cylinder are given by $z = 0$ and $z = H$, and the sidewall is located at $r = b$. The ring source is at $(r = a, z = h)$, while the ring sink lies in the bottom corner of the cylinder $(r = b, z = 0)$.

In the annular region $a < r < b$, $0 < z < H$ the flow at some distance from the solid walls is not affected by viscosity, and is governed by a geostrophic balance of forces:

$$2\boldsymbol{k} \times \boldsymbol{v}_{\mathrm{I}} = -\nabla p_{\mathrm{I}}, \quad \nabla \cdot \boldsymbol{v}_{\mathrm{I}} = 0 \tag{2.4}$$

(the subscript I denotes geostrophic interior motion). Ekman layers arise at the horizontal boundaries at $z = 0$ and $z = H$ in order to adjust the flow to relative rest. Because there is no differential rotation between the horizontal disks, the suction conditions imposed by these Ekman layers require that $w_{\mathrm{I}} = 0$. As can be seen from the azimuthal component of the momentum equation (2.4), radial motion is absent in axisymmetric interior configurations. This implies that the radial $O(E^{1/2})$ transport $Q = 2\pi a\hat{Q}_a = 2\pi b\hat{Q}_b$ from the source to the sink must entirely take place via the Ekman layers at both horizontal disks. Since the local (i.e. per unit length of circumference) radial transport in each of these Ekman layers measures $-\frac{1}{2}v_{\mathrm{I}}(r)E^{1/2}$, it follows that

$$v_{\mathrm{I}}(r) = -\frac{Q}{2\pi r} = -\frac{a\hat{Q}_a}{r}, \quad u_{\mathrm{I}} = w_{\mathrm{I}} = 0. \tag{2.5}$$

At the sidewall $(r = b)$ the azimuthal interior velocity is brought to relative rest by a Stewartson layer, consisting of an outer $E^{1/4}$ layer and a thin inner $E^{1/3}$ layer. The structure of such a boundary layer is wellknown, and for the situation with a singularity (source or sink) in the top or bottom corner it has been discussed by Conlisk and Walker [9]. Besides matching the interior swirl velocity, the Stewartson layer also carries a net $O(E^{1/2})$ transport of magnitude $\frac{1}{2}\hat{Q}_b$ downwards from the upper Ekman layer to the sink.

3. The shear-layer structure

The occurrence of a Stewartson layer at $r = a$ is apparent from two reasons. First, a net $O(E^{1/2})$ transport towards the Ekman layers at $z = 0$ and $z = H$ must be accomplished; such a vertical flux is not possible in the interior region, and has necessarily to be carried by a Stewartson layer. Secondly, the azimuthal interior flow driven by the Ekman layers must be matched smoothly to the zero-flow region $r < a$, $0 < z < H$. As will be shown in the sequel this requires the presence of both an $E^{1/4}$ layer and an inner $E^{1/3}$ layer, with the former performing the $O(1)$ matching and producing some $O(E^{1/2})$ flux, and the latter smoothing discontinuities in the $E^{1/4}$ layer and completing the $O(E^{1/2})$ transport to the Ekman layers.

By considering the total $O(E^{1/2})$ transport across a horizontal plane at arbitrary height z, including contributions from the sidewall layer ($T_b = -\frac{1}{2}\hat{Q}_b$) and the Stewartson layer at the source (T_a), one derives

$$\begin{aligned} 2\pi a T_a(z) + 2\pi b T_b &= -Q, \quad 0 < z < h \\ &= 0, \quad h < z < H \end{aligned} \tag{3.1}$$

or equivalently

$$\begin{aligned} T_a(z) &= -\tfrac{1}{2}\hat{Q}_a, \quad 0 < z < h \\ &= +\tfrac{1}{2}\hat{Q}_a, \quad h < z < H. \end{aligned} \tag{3.2}$$

This transport condition will be referred to in the analyses of the $E^{1/4}$ layer and the $E^{1/3}$ layer.

3.1. The $E^{1/4}$ layer

In the $E^{1/4}$ layer the velocity components and the pressure are expanded in powers of $E^{1/4}$:

$$(u, v, w, p) = \sum_{j=0}^{\infty} (E^{1/2}U, V, E^{1/4}W, E^{1/4}P)E^{j/4},$$

and substitution into the linear equations (2.1) yields for the leading terms

$$U_z = V_z = W_{zz} = 0, \quad V_{\xi\xi\xi} + 2W_z = 0, \quad 2U = V_{\xi\xi} \tag{3.3}$$

where $\xi = (r - a)E^{-1/4}$ is the stretched radial coordinate. The Ekman suction conditions

$$W(z=0) = \tfrac{1}{2}V_\xi, \quad W(z=H) = -\tfrac{1}{2}V_\xi$$

enable one to derive from (3.3)

$$V_{\xi\xi\xi} - \frac{2}{H}V = 0,$$

which has the following general solutions:

$$\begin{aligned} V^+(\xi) &= A^+ + B^+ \exp(-s\xi), \quad \xi \geqslant 0 \\ V^-(\xi) &= A^- + B^- \exp(s\xi), \quad \xi \leqslant 0, \end{aligned} \tag{3.4}$$

where $s = \sqrt{2/H}$. The constants $A^\pm$, $B^\pm$ can be determined by matching to the interiors on either side ($\xi \to \pm\infty$) and by requiring continuity of V and V_ξ at $\xi = 0$; then it follows from (3.3):

$$V^+(\xi) = -\tfrac{1}{2}\hat{Q}_a[2 - \exp(-s\xi)], \quad V^-(\xi) = -\tfrac{1}{2}\hat{Q}_a \exp(s\xi), \tag{3.5a}$$

$$W^\pm(\xi, z) = -\tfrac{1}{2}s\hat{Q}_a\left(\tfrac{1}{2} - \frac{z}{H}\right)\exp(-s|\xi|), \tag{3.5b}$$

$$U^\pm(\xi) = \operatorname{sgn}(\xi)\frac{\hat{Q}_a}{2H}\exp(-s|\xi|). \tag{3.5c}$$

Inspection of these results shows that $V_{\xi\xi}$, and therefore U as well, are discontinuous at $\xi = 0$. These discontinuities have to be smoothed by the inner $E^{1/3}$ layer at $\xi = 0$.

The vertical $O(E^{1/2})$ transport in the $E^{1/4}$ layer is found by integrating W over the layer thickness, yielding

$$\int_{-\infty}^{+\infty} W(\xi, z)\mathrm{d}\xi = -\left(\tfrac{1}{2} - \frac{z}{H}\right)\hat{Q}_a. \tag{3.6}$$

Although this vertical transport is correctly balanced by the radial Ekman fluxes at $z=0$ and $z=H$, an additional contribution is needed to satisfy the flux condition (3.2) on $0<z<H$. This contribution measures

$$\begin{aligned} \tilde{T}(z) &= -\frac{z}{H}\hat{Q}_a, \qquad 0<z<h \\ &= \left(1-\frac{z}{H}\right)\hat{Q}_a, \quad h<z<H \end{aligned} \tag{3.7}$$

and has to be produced by the $E^{1/3}$ layer. From the U solution (3.5c) it can be seen that the $E^{1/4}$ layer is fed radially by the $E^{1/3}$ layer at $\xi=0$:

$$\int_0^H U^+(\xi=0)\mathrm{d}z = -\int_0^H U^-(\xi=0)\mathrm{d}z = \tfrac{1}{2}\hat{Q}_a. \tag{3.8}$$

The incoming fluxes at $\xi=0^\pm$ are uniform in z and are, according to (3.6), equally distributed to the Ekman layers at the horizontal disks (each Ekman layer receives a flux of magnitude $\frac{1}{2}\hat{Q}_a$), yielding a symmetrical $O(E^{1/2})$ circulation pattern in the $E^{1/4}$ layer. In conclusion, the main task of the $E^{1/3}$ layer lies in the distribution of the injected fluid in such a way that the $E^{1/4}$ layer receives a uniform flux given by (3.8) and that a vertical flux is produced as required by (3.7). It will become clear in the next section that these requirements are consistent with the smoothing of the discontinuities in U and $V_{\xi\xi}$ as mentioned above.

3.2. The $E^{1/3}$ layer

The velocity components and the pressure in the $E^{1/3}$ layer are expanded as

$$(u,v,w,p) = \sum_{j=0}^{\infty}\left(E^{1/3}\tilde{u}_j, \tilde{v}_j, \tilde{w}_j, E^{1/3}\tilde{p}_j\right)E^{j/12},$$

and substitution into (2.1) yields for each field j, after elimination of the pressure,

$$2\tilde{v}_z = \tilde{w}_{\eta\eta\eta}, \quad 2\tilde{w}_z = -\tilde{v}_{\eta\eta\eta}, \quad 2\tilde{u} = \tilde{v}_{\eta\eta} \tag{3.9}$$

with $\eta=(r-a)E^{-1/3}$. With regard to the orders of magnitude, it turns out that the smoothing of the jump discontinuities in U and $V_{\xi\xi}$, and the production of vertical $O(E^{1/2})$ transport must be carried out by the $j=2$ field. The boundary conditions for this $j=2$ problem are

$$\eta \to \pm\infty: \quad \tilde{v}_2, \tilde{w}_2 \to 0; \tag{3.10a}$$

$$\begin{aligned} \eta=0: \quad & \tilde{v}_2, \tilde{w}_2, \frac{\partial \tilde{v}_2}{\partial\eta}, \frac{\partial \tilde{w}_2}{\partial\eta} \quad \text{continuous,} \\ & \left.\frac{\partial^2\tilde{v}_2}{\partial\eta^2}\right|_{0^+} + V^+_{\xi\xi}|_0 = \left.\frac{\partial^2\tilde{v}_2}{\partial\eta^2}\right|_{0^-} + V^-_{\xi\xi}|_0; \end{aligned} \tag{3.10b}$$

$$z=0, H: \quad \tilde{w}_2 = 0; \tag{3.10c}$$

$$\begin{aligned} 0<z<h: \quad & \int_{-\infty}^{+\infty} \tilde{w}_2 \mathrm{d}\eta &= -\frac{z}{H}\hat{Q}_a; \\ h<z<H: \quad & &= \left(1-\frac{z}{H}\right)\hat{Q}_a. \end{aligned} \tag{3.10d}$$

As in Conlisk and Walker's study [9] on the injection through thin slots in the sidewall, the ring source (which is assumed to have an infinitesimally small cross-section) is enveloped by a ring-shaped region of cross-sectional dimensions $O(E^{1/2} \times E^{1/2})$. The injected fluid is rearranged in this region, and erupts vertically upwards and downwards into the adjacent $E^{1/3}$ layer. On the scale of the $E^{1/3}$ layer this eruption appears as a $\delta(\eta)$-singularity in the vertical velocity, which is merely a consequence of the fact that $E^{1/2} \ll E^{1/3}$, as E is a small number. Quite generally, one can write

$$\tilde{w}_2(z=h^-) = C^-\delta(\eta), \quad \tilde{w}_2(z=h^+) = C^+\delta(\eta), \tag{3.11}$$

where the singularity strengths $C^\pm$ are linked by mass conservation, viz. $C^+ - C^- = \hat{Q}_a$. Their individual values follow from the conditions (3.10d) for the vertical transport in the $E^{1/3}$ layer, yielding

$$C^- = -\frac{h}{H}\hat{Q}_a, \quad C^+ = \left(1-\frac{h}{H}\right)\hat{Q}_a. \tag{3.12}$$

The $(\tilde{v}_2, \tilde{w}_2)$ problem (3.9) – (3.12) can be solved by writing the general solutions as Fourier series:

$$\tilde{w}_2^\pm = \sum_{n=1}^{\infty} \left[a_n^\pm \mathrm{e}^{-\gamma_n|\eta|} + b_n^\pm \mathrm{e}^{\omega\gamma_n|\eta|} + c_n^\pm \mathrm{e}^{\omega^2\gamma_n|\eta|} \right] \sin\frac{n\pi z}{H}, \tag{3.13a}$$

$$\tilde{v}_2^\pm = \pm \sum_{n=1}^{\infty} \left[a_n^\pm \mathrm{e}^{-\gamma_n|\eta|} - b_n^\pm \mathrm{e}^{\omega\gamma_n|\eta|} - c_n^\pm \mathrm{e}^{\omega^2\gamma_n|\eta|} \right] \cos\frac{n\pi z}{H}, \tag{3.13b}$$

with $\omega = -\frac{1}{2} + \frac{1}{2}\mathrm{i}\sqrt{3}$, and by applying the boundary conditions (3.10b) in order to determine the coefficients $a_n^\pm$, $b_n^\pm$ and $c_n^\pm$. However, the transport condition (3.10d) offers an important clue, as by using the series property (see e.g [10])

$$\begin{aligned} \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\cos\frac{n\pi h}{H}\sin\frac{n\pi z}{H} &= -\frac{z}{H}, \quad 0<z<h \\ &= 1-\frac{z}{H}, \quad h<z<H \end{aligned} \tag{3.14}$$

it can be written in the following form:

$$\int_{-\infty}^{+\infty} \tilde{w}_2 \mathrm{d}\eta = \frac{2\hat{Q}_a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\frac{n\pi h}{H}\sin\frac{n\pi z}{H}. \tag{3.15}$$

Now it is straightforward to calculate the coefficients, yielding

$$\tilde{w}_2(\eta, z) = \sum_{n=1}^{\infty} A_n\left(\phi_3 + \phi_2 + \phi_1\sqrt{3}\right) \cos\frac{n\pi h}{H} \sin\frac{n\pi z}{H},$$
$$\tilde{v}_2(\eta, z) = \operatorname{sgn}(\eta) \sum_{n=1}^{\infty} A_n\left(\phi_3 - \phi_2 - \phi_1\sqrt{3}\right) \cos\frac{n\pi h}{H} \cos\frac{n\pi z}{H} \tag{3.16}$$

with

$$A_n = \frac{\hat{Q}_a}{3n\pi}\gamma_n, \quad \gamma_n = (2n\pi/H)^{1/3}$$

and

$$\phi_1(\gamma_n;\, \eta) = \mathrm{e}^{-(1/2)\gamma_n|\eta|} \sin\tfrac{1}{2}\gamma_n|\eta|\sqrt{3},$$
$$\phi_2(\gamma_n;\, \eta) = \mathrm{e}^{-(1/2)\gamma_n|\eta|} \cos\tfrac{1}{2}\gamma_n\sqrt{3}, \tag{3.17}$$
$$\phi_3(\gamma_n;\, \eta) = \mathrm{e}^{-\gamma_n|\eta|}.$$

The solution for the radial velocity follows immediately from (3.9c), giving:

$$\tilde{u}_2(\eta, z) = \operatorname{sgn}(\eta) \sum_{n=1}^{\infty} \tfrac{1}{2}\gamma_n^2 A_n (2\phi_2 + \phi_3) \cos\frac{n\pi h}{H} \cos\frac{n\pi z}{H}. \tag{3.18}$$

As can be seen by differentiating (3.14) with respect to z, it appears that

$$\tilde{u}_2(\eta = 0^{\pm}) = -\operatorname{sgn}(\eta)\frac{\hat{Q}_a}{2H}, \quad 0 < z < H, \tag{3.19}$$

which means that the total radial $O(E^{1/2})$ velocity $\tilde{u}_2 + U$ is now continuous across η, $\xi = 0$. This implies that $\tilde{v}_{\eta\eta} + V_{\xi\xi}$ is continuous as well.

Finally, it is worth mentioning that the structure of the vertical $O(E^{1/2})$ transport in the $E^{1/3}$ layer is closely related to the occurrence of a jump discontinuity in $\tilde{v}_{\eta\eta}$. This can be illustrated by integration of the governing equation (3.9b) with respect to η, which yields

$$\frac{\partial}{\partial z}\int_{-\infty}^{+\infty} \tilde{w}_2 \mathrm{d}\eta = -\tfrac{1}{2}\int_{-\infty}^{+\infty} \frac{\partial^3 \tilde{v}_2}{\partial \eta^3}\mathrm{d}\eta = -\tfrac{1}{2}\left[\left.\frac{\partial^2 \tilde{v}_2}{\partial \eta^2}\right|_{-\infty}^{0^-} + \left.\frac{\partial^2 \tilde{v}_2}{\partial \eta^2}\right|_{0^+}^{+\infty}\right].$$

From the continuity requirement (3.10b) it follows that

$$\frac{\partial}{\partial z}\int_{-\infty}^{+\infty} \tilde{w}_2 \mathrm{d}\eta = -\frac{\hat{Q}_a}{H}. \tag{3.20}$$

Integration with respect to z on the separate intervals $0 < z < h$ and $h < z < H$, and applying $\tilde{w}_2 = 0$ at the boundaries $z = 0$, H results in the expression (3.10d) for the vertical flux.

This completes the analysis of the $O(E^{1/2})$ circulation in the Stewartson layer at $r = a$. Returning to the $E^{1/4}$-layer solutions (3.5), it appears that, though W is continuous across $\xi = 0$, its ξ-derivative contains a jump discontinuity there. In view of the orders of magnitude, this discontinuity must be smoothed by the $j = 4$ field of the $E^{1/3}$ layer, which can be analysed in the usual way. The analysis is slightly complicated due to the more involved form of the Ekman suction conditions,

$$\tilde{w}_4(z=0) = \tfrac{1}{2}\frac{\partial \tilde{v}_2}{\partial \eta}\bigg|_{z=0}, \quad \tilde{w}_4(z=H) = -\tfrac{1}{2}\frac{\partial \tilde{v}_2}{\partial \eta}\bigg|_{z=H}. \tag{3.21}$$

A similar problem has been discussed in [11], however, and the analysis of the $j = 4$ field is therefore omitted here.

4. Discussion

In the previous section the structure of a detached Stewartson layer has been analysed for the situation where fluid is emitted from a ring source at an arbitrary distance from the horizontal flow boundaries. With regard to the $O(E^{1/2})$ circulation the $E^{1/4}$ layer appears to play a passive part, and its presence is only required for the matching of the interior azimuthal velocities on either side. Details of the source configuration are only noticeable in the inner $E^{1/3}$ layer. The fluid emitted from the source is rearranged in a small region of cross-sectional dimensions $O(E^{1/2} \times E^{1/2})$ before it erupts vertically into the $E^{1/3}$ layer. The singularity strengths associated with these eruptions can be found from general transport arguments, and appear to depend on the distances between the source and the end caps of the cylinder, see (3.12). It is easy to verify that the singularity strengths (3.12) show a gradual transition to values associated with injection through ring sources at the bottom or the top disk, as $h \to 0$ or $h \to H$, respectively (cf. Hashimoto [4]). This is in contrast with Conlisk and Walker's [9] treatment of the singularity strengths associated with injection through a thin slot in the sidewall: they assume symmetry by requiring the upward and downward directed singularities to be of equal strength. In fact this is only appropriate for the special case where the source is positioned at equal distances from the end caps. Fortunately their subsequent results do not seem to be affected by the symmetry assumption. This is easily verified by using the solution procedure applied in Section 3 of the present paper (it is worth noting that this solution technique is much less involved than Conlisk and Walker's technique of introducing mirror images of the singularities in order to account for the impermeability of the end caps). According to the technique described in Section 3, the velocities $\tilde{v}_2$ and $\tilde{w}_2$ are written as general Fourier series of the form (3.13); the coefficients are determined by requiring no-slip at the sidewall and by applying the vertical-transport condition, which in Conlisk and Walker's problem takes a form similar to (3.15). Then it follows after a few lines of algebra that their results are correct.

The analysis of the detached Stewartson-layer structure due to a single source (Section 3) allows extension to multiple sources and sinks positioned at different heights. Assume there is an array of N sources at radius $r = a$ located at arbitrary heights $z = h_i$ ($i = 1, 2, \ldots, N$), and with strengths Q_i such that

$$\sum_{i=1}^{N} Q_i/2\pi a = \sum_{i=1}^{N} \hat{Q}_{ai} = \hat{Q}_a. \tag{4.1}$$

By considering the vertical $O(E^{1/2})$ flux across a horizontal plane at arbitrary height z, including contributions from the Stewartson layer at the sidewall ($r=b$) and the $E^{1/4}$ layer at $r=a$, one derives for the vertical $O(E^{1/2})$ transport $\tilde{T}$ in the $E^{1/3}$ layer:

$$\tilde{T}(z) = -\frac{z}{H}\hat{Q}_a + \sum_{i=1}^{k} \hat{Q}_{ai}, \quad h_k < z < h_{k+1} \quad (k=0, 1, \ldots, N) \tag{4.2}$$

where $h_0 = 0$, $h_{N+1} = H$. Again, the general solutions for $\tilde{v}_2$ and $\tilde{w}_2$ are written as Fourier series (3.13), and the coefficients are determined by applying the boundary conditions (3.10b) and the transport condition (4.2). Inspection of (4.2) shows that $\tilde{T}(z)$ consists of a term linear in z and a number of "steps" associated with the sources. The first term can be written as (see e.g. [10])

$$f(z) = 2\hat{Q}_a \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin\frac{n\pi z}{H}, \quad 0 < z < H, \tag{4.3}$$

and a sine-series representation of a step function that satisfies

$$\begin{aligned} g_i(z) &= 0, & 0 < z < h_i \\ &= \hat{Q}_{ai}, & h_i < z < H \end{aligned}$$

is found by elementary Fourier theory, yielding

$$g_i(z) = 2\hat{Q}_{ai} \sum_{n=1}^{\infty} \frac{1}{n\pi} \left\{ \cos\frac{n\pi h_i}{H} - (-1)^n \right\} \sin\frac{n\pi z}{H}, \quad 0 < z < H. \tag{4.4}$$

The transport condition (4.2) is then concisely formulated as

$$\begin{aligned} \tilde{T}(z) &= f(z) + \sum_{i=1}^{N} g_i(z) \\ &= 2 \sum_{i=1}^{N} \hat{Q}_{ai} \sum_{n=1}^{\infty} \frac{1}{n\pi} \cos\frac{n\pi h_i}{H} \sin\frac{n\pi z}{H}, \quad 0 < z < H, \end{aligned} \tag{4.5}$$

and can be used directly to determine the coefficients in the general solutions (3.13) for $\tilde{v}_2$ and $\tilde{w}_2$. It is obvious that negative values of $\hat{Q}_{ai}$ are also allowed for, provided that condition (4.1) is still satisfied. This means that the analysis is also applicable to any combination of multiple sources and sinks of arbitrary strength.

Similarly, the extension can be made to a distributed line source. If the fluid is emitted uniformly from a distributed source of height $2d$ centred at $z = h$, the vertical $O(E^{1/2})$ flux $\tilde{T}(z)$ in the $E^{1/3}$ layer is calculated as in the multiple-source configuration, yielding

$$\begin{aligned} \tilde{T}(z) &= -\frac{z}{H}\hat{Q}_a, & 0 < z < h-d, \\ &= \left(1 - \frac{z}{H}\right)\hat{Q}_a + \frac{\hat{Q}_a}{2d}(z-h-d), & h-d < z < h+d, \\ &= \left(1 - \frac{z}{H}\right)\hat{Q}_a, & h+d < z < H. \end{aligned} \tag{4.6}$$

The Fourier sine-series representation of this function is

$$\tilde{T}(z) = \frac{2\hat{Q}_a}{Hd} \sum_{n=1}^{\infty} \left(\frac{H}{n\pi}\right)^2 \sin\frac{n\pi d}{H} \cos\frac{n\pi h}{H} \sin\frac{n\pi z}{H}, \quad 0 < z < H, \tag{4.7}$$

and the $\tilde{v}_2$, $\tilde{w}_2$ solutions can be determined as before.

Due to the insight obtained from previous work ([4], [9]) and from the present study, it is now possible to analyse the flow driven by any axisymmetric arrangement of single, multiple or distributed sources and sinks located at the sidewall, at the end caps and/or at arbitrary positions at some distance from the boundaries.

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